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LETTER TO THE EDITOR

Dilute Ising model on fractal lattices

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Abstract. We study dilute Ising models on a family of infinitely ramified exact fractals whose fractal dimension varies from 1 to 2. We calculate the phase diagram and critical exponents as a function of the fractal dimensionality within Migdal's approximation.

Phase transitions and percolation on fractal lattices have recently been a subject of intensive investigation (Mandelbrot 1982, Gefen *et al* 1980, 1983, 1984, Havlin *et al* 1983). Infinitely ramified fractals have been shown to exhibit second-order phase transitions at finite temperature and non-trivial percolation thresholds.

In this letter we study dilute Ising models on a family of infinitely ramified fractals whose fractal dimension varies from 1 to 2. This problem has been studied extensively on regular euclidean lattices and temperature–concentration phase diagrams have been calculated for these systems (Stinchcombe 1983). On the family of fractals which will be described below we found, within Migdal's approximation (Migdal 1975), similar phase diagrams. Phase diagrams and critical exponents have been calculated as a function of fractal dimensionality. These fractals represent a physical model to study the problem of dimension d approaching unity from above (Kirkpatrick 1977, Stauffer and Jayaprakash 1978). The fractal family we have studied is shown in figure 1. These fractals are infinitely ramified (Gefen *et al* 1984, Havlin *et al* 1983). The fractal dimensionality as a function of the scaling factor b is given by

$$d_{\mathrm{f}}=\frac{\ln(b^2-(b-2)^2)}{\ln b}=\frac{\ln 4(b-1)}{\ln b}. \tag{1}$$

As b varies from 2 to infinity d_{f} varies from 2 to 1. In the vicinity of $d_{\mathrm{f}}=1$ ($b=\infty$) we have

$$\varepsilon=d_{\mathrm{f}}-1=\ln 4/\ln b. \tag{2}$$

For a one-dimensional pure Ising model the exact recursion relation is

$$\tanh \tilde{K}=(\tanh K)^b, \tag{3}$$

where b is the scaling factor, $K=\beta J$ the reduced nearest-neighbour interaction and $\tilde{K}$ the renormalised interaction.

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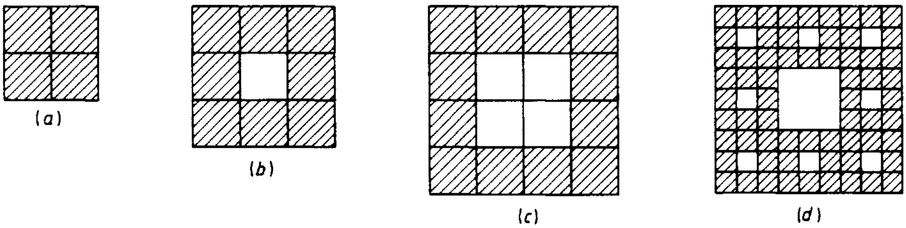

Figure 1. (*a*), (*b*) and (*c*) are the generators of the fractal family for $b=2$, 3 and 4 respectively. (*d*) represents the second iteration for $b=3$.

For a two-dimensional square lattice the Migdal recursion relation is

$$K' = b\tilde{K} = b \tanh^{-1}(\tanh^b K). \tag{4}$$

For the fractal family described above instead of having b paths we have only $b-(b-2)$ paths and consequently the approximate Migdal relation is

$$K' = 2\tilde{K} = 2\tanh^{-1}(\tanh^b K) \tag{5}$$

or

$$t' = 2t^b/(1+t^{2b}) \tag{6}$$

where $t = \tanh K$ and $t' = \tanh K'$. The recursion relation (6) has three fixed points

$$t=0, \qquad t=1 \qquad \text{and} \qquad t=t_c(b).$$

The first two are stable, and correspond respectively to the para- and ferromagnetic phases; the third is unstable and corresponds to the para-ferromagnetic and second-order phase transition. If the b interactions of each of the two paths are random variables, equation (6) becomes

$$t' = \frac{t_{11}t_{12}\ldots t_{1n} + t_{21}t_{22}\ldots t_{2n}}{1 + t_{11}t_{12}\ldots t_{1n}t_{21}\ldots t_{2n}} \tag{7}$$

where $t_{ij}(i=1,2,\ldots,b)$ are independent random variables defined by $t_{ij} = \tanh K_{ij}$.

In the case of dilution the probability distribution for the random variable t is

$$P(t) = (1-p)\delta(t) + p\delta(t-t_0). \tag{8}$$

A simple analysis yields the renormalised probability distribution

$$P'(t') = (1-2p^b+p^{2b})\delta(t') + 2(p^b-p^{2b})\delta(t'-t_0^b) + p^{2b}\delta(t'-2t_0^b/(t+t_0^{2b})). \tag{9}$$

Equation (9) is not of the form of equation (8) and we used the two-peak approximation (Stinchcombe 1983) to obtain the recursion relations

$$p' = 2p^b - p^{2b} \tag{10a}$$

$$p't' = p^{2b}2t^b/(1+t^{2b}) + 2(p^b - p^{2b})t^b \tag{10b}$$

where index 0 has been dropped. Equation (10*a*) is the same as in Havlin *et al* (1983).

The flow diagram which corresponds to equations (10) is shown in figure 2 in the case $b=3$.

In the limit $d_f \to 1$ $(b\to\infty)$

(i) $T_c \sim 1/\ln b \sim \varepsilon$

(ii) $P_c \sim 1 - 1/b^2$.

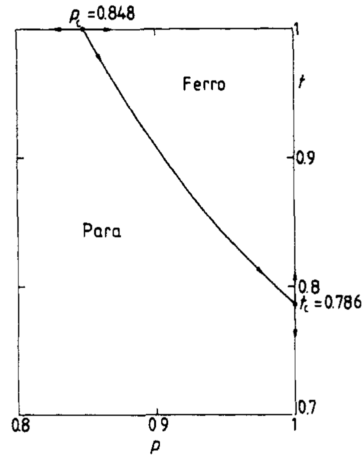

Figure 2. Flow diagram for $b = 3$.

We recover the fact that the lower critical dimensionality for the Ising model is equal to 1.

In the vicinity of the pure fixed point (t_c) the critical exponent is

$$1/\nu = y = \ln 2/\ln b = \tfrac{1}{2}\varepsilon. \tag{11}$$

In the vicinity of the percolation fixed point (p_c) the critical exponents are

$$\begin{aligned} 1/\nu_p &= y_p = \ln 2/\ln b = \tfrac{1}{2}\varepsilon \\ 1/\nu_t &= y_t = \ln 2/\ln b = \tfrac{1}{2}\varepsilon \end{aligned} \tag{12}$$

and consequently the crossover exponent φ is equal to 1.

The critical exponents in the vicinity of $d_f = 1$ are proportional to $\varepsilon = d_f - 1$ as expected (Migdal 1975, Kirkpatrick 1977). The proportionality constant depends upon the family of fractals used to model the approach of $d_f = 1$ (Gefen *et al* 1983).

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